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R.A.C. Correa  
*Dartmouth College*

A. de Souza Dutra  
*Universidade Estadual Paulista em Guaratinguetá*

M. Gleiser  
*Dartmouth College*

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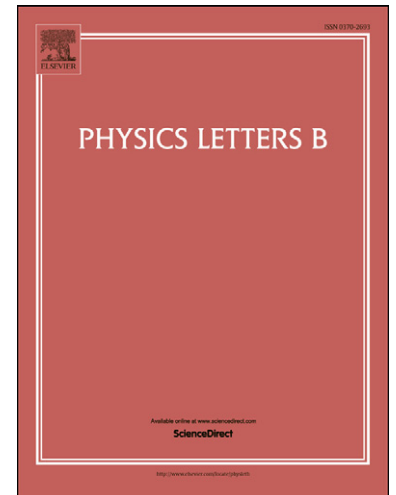
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# Information-Entropic Measure of Energy-Degenerate Kinks in Two-Field Models

R. A. C. Correa<sup>1, 2, \*</sup>, A. de Souza Dutra<sup>1, †</sup>, and M. Gleiser<sup>2, ‡</sup>  
<sup>1</sup>UNESP-Campus de Guaratinguetá-DFQ, Av. Dr. Ariberto Pereira Cunha,  
 333 C.P. 205 12516-410 Guaratinguetá SP Brasil

<sup>2</sup>Center for Cosmic Origins and Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755, USA  
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We investigate the existence and properties of kink-like solitons in a class of models with two interacting scalar fields. In particular, we focus on models that display both double and single-kink solutions, treatable analytically using the Bogomol'nyi-Prasad-Sommerfield bound (BPS). Such models are of interest in applications that include Skyrmions and various superstring-motivated theories. Exploring a region of parameter space where the energy for very different spatially-bound configurations is degenerate, we show that a newly-proposed momentum-space entropic measure called Configurational Entropy (CE) can distinguish between such energy-degenerate spatial profiles. This information-theoretic measure of spatial complexity provides a complementary perspective to situations where strictly energy-based arguments are inconclusive.

Keywords: Entropy, nonlinear, bloch walls, kinks, lumps

## 1. INTRODUCTION

Since the Scottish channel engineer John Scott Russell first discovered the existence of solitary waves in 1834 [1] and, in particular, since the 1960s and 70s [2]-[7], the study of nonlinear solutions of PDEs that preserve their spatial profile has attracted much interest in many areas of physics, such as in cosmology [8], field theory [9, 10], condensed matter physics [11], and others [12]. In high-energy physics, solitons [10]-[13] are generally known as solutions of nonlinear field equations whose energy density is localized in space. Certain soliton solutions, as in the case of sine-Gordon kinks [13], have the interesting feature of keeping their shape unaltered after scattering with other solitons. (Here, we will use “soliton” to characterize solutions with localized energy-density, even if many may not maintain their spatial profile after scattering.)

Nowadays, the properties of nonlinear configurations are well understood in a wide class of models with or without spontaneous symmetry breaking, and with or without a nontrivial topological vacuum structure. Of particular interest to us here are kinks, non-dissipative solutions with an associated topological charge. Kink configurations arise in (1+1)-dimensional field theories when the scalar field potential has two or more degenerate minima. A well-known example is the  $\phi^4$ -kink, also called the  $Z_2$  kink [9, 14]. In this case, a single real scalar field  $\phi$  interpolates between the two degenerate minima of the potential.

A powerful insight to solve nonlinear problems analytically was introduced by Bogomol'nyi [15], Prasad and Sommerfield [16]. Known as the BPS bound, it is based on obtaining a first-order differential equation from the

energy functional. With this method, it is possible to find solutions that minimize the energy of the configuration while ensuring their stability. A large variety of models in the literature use the BPS approach, such as solutions found in Skyrme models [17, 18], monopoles [19, 20], supersymmetric black holes [21], supergravity [22], and  $K$ -field theories [23].

A few decades ago, it was shown that it is possible to find kink-like solutions for certain coupled scalar field theories in (1+1)-dimensional models. Presented by Rajaraman, the approach is based on a “trial and error” method which leads to important particular solutions [24]. Bazeia and collaborators [25] showed that solutions of certain second-order differential systems with two or more scalar fields can be mapped into a corresponding set of first-order nonlinear differential equations, so that one can obtain the general solution of the system [26].

In an apparently disconnected topic, in 1948 Shannon defined the entropy of a data string as a measure of how much information is needed to characterize it in a transmission: the more information is needed for a reliable transmission, the higher the entropy. Inspired by Shannon, Gleiser and Stamatopoulos (GS) recently proposed a measure of complexity of a localized mathematical function [27]. GS proposed that the Fourier modes of square-integrable, bounded mathematical functions can be used to construct a measure of what they called configurational entropy (CE): a configuration consisting of a single mode has zero CE (a single wave in space), while one where all modes contribute with equal weight has maximal CE. To apply such ideas to physical models, GS used the energy density of a given spatially-localized field configuration, found from the solution-exact or approximate-of the related PDE. Of importance in what follows, GS pointed out that the configurational entropy can be used to choose the best-fitting trial function in situations where their energies are degenerate. More generally, the approach presented in [27] has been recently used to study the nonequilibrium dynamics of spontaneous symmetry breaking [28], to obtain a stabil-

\*E-mail: fis04132@gmail.com

†E-mail: dutra@gmail.com

‡E-mail: mgleiser@dartmouth.edu

ity bound for compact astrophysical objects [29], and to investigate the emergence of localized objects during inflationary preheating [30].

In the present work we will compute the configurational entropy of some classes of models with two interacting scalar fields [24–26, 31]. These models admit a variety of kink-like solutions, and have been shown to give rise to bags, junctions, and networks of BPS and non-BPS defects [32]. In particular, we will explore analytical solutions that are energy-degenerate but quite distinct in their spatial profiles. We will show that the CE can be used to distinguish between such configurations, adding a new information-theoretic perspective to the study of BPS solitons and other nonlinear localized configurations.

Section II introduces the model and its analytical solutions. Section III reviews the configurational entropy measure for spatially localized solutions. In section IV we compute the configurational entropy for two-field BPS solitons and show how it can be used to distinguish between energy-degenerate configurations. In section V we present our conclusions and final remarks.

## 2. INTERACTING SCALAR FIELD MODEL AND ITS SOLUTIONS

Consider a (1+1)-dimensional model with two interacting real scalar fields described by the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi)^2 + \frac{1}{2}(\partial_\nu \chi)^2 - V(\phi, \chi), \quad (1)$$

where  $V(\phi, \chi)$  is the potential. We use units with  $c = \hbar = 1$  and metric  $\eta_{\nu\beta} = \text{diag}(1, -1)$  with coordinates  $x^\nu = (t, x)$ .

The potential  $V(\phi, \chi)$  can be represented in terms of a superpotential  $W(\phi, \chi)$  as

$$V(\phi, \chi) = \frac{1}{2} \left[ \left( \frac{\partial W(\phi, \chi)}{\partial \phi} \right)^2 + \left( \frac{\partial W(\phi, \chi)}{\partial \chi} \right)^2 \right]. \quad (2)$$

This representation includes supersymmetric models that generate distinct domain walls and topological solitons [33]–[35].

From the Lagrangian density (1) and the definition of the superpotential (2), the classical Euler–Lagrange equations of the static field configurations  $\phi = \phi(x)$  and  $\chi = \chi(x)$  are given by

$$\frac{d^2 \phi}{dx^2} = W_\phi W_{\phi\phi} + W_\chi W_{\chi\phi}, \quad (3)$$

$$\frac{d^2 \chi}{dx^2} = W_\chi W_{\chi\chi} + W_\phi W_{\phi\chi}, \quad (4)$$

where the subscripts denote derivatives with respect to the two fields. The energy functional of the static field

configurations can be calculated as

$$E_{BPS} = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \left( \frac{d\phi}{dx} \right)^2 + \left( \frac{d\chi}{dx} \right)^2 + W_\phi^2 + W_\chi^2 \right], \quad (5)$$

where  $W_\phi \equiv \frac{\partial W(\phi, \chi)}{\partial \phi}$  and  $W_\chi \equiv \frac{\partial W(\phi, \chi)}{\partial \chi}$ . The above functional energy can be easily rewritten in the following form

$$E_{BPS} = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \left( \frac{d\phi}{dx} - W_\phi \right)^2 + \left( \frac{d\chi}{dx} - W_\chi \right)^2 + 2 \left( W_\phi \frac{d\phi}{dx} + W_\chi \frac{d\chi}{dx} \right) \right]. \quad (6)$$

As a consequence, the solutions with minimal energy of the second-order differential equations for the static solutions can be found from the following two first-order equations

$$\frac{d\phi}{dx} = W_\phi, \quad \text{and} \quad \frac{d\chi}{dx} = W_\chi. \quad (7)$$

The energy  $E_{BPS}$ , which is called BPS energy, is written as

$$E_{BPS} = |W(\phi_j, \chi_j) - W(\phi_i, \chi_i)|, \quad (8)$$

where  $\phi_i$  and  $\chi_i$  denote the  $i$ th vacuum state of the model.

Following Ref. [26], it is possible from (7) to formally write the equation

$$\frac{d\phi}{W_\phi} = dx = \frac{d\chi}{W_\chi}, \quad (9)$$

which leads to

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi}. \quad (10)$$

The above equation is a nonlinear differential equation relating the scalar fields of the model so that  $\phi = \phi(\chi)$ . Once this function is known, equations (7) become uncoupled and can be solved.

Considering the application below, we now review the model studied in Refs. [25, 26, 32], used for modeling a great number of systems [32]–[41], whose superpotential is given by

$$W(\phi, \chi) = -\lambda\phi + \frac{\lambda}{3}\phi^3 + \mu\phi\chi^2, \quad (11)$$

where  $\lambda$  and  $\mu$  are real and positive dimensionless coupling constants. The potential  $V(\phi, \chi)$  of the model with the above superpotential is given by

$$V(\phi, \chi) = \frac{1}{2} \left[ \lambda^2 + \lambda^2 \phi^2 (\phi^2 - 2) + \mu^2 \chi^2 \left( \chi^2 - \frac{2\lambda}{\mu} \right) + 2\mu^2 \left( \frac{\lambda}{\mu} + 2 \right) \phi^2 \chi^2 \right]. \quad (12)$$

For  $\lambda/\mu > 0$  the model has four supersymmetric minima  $(\phi, \chi)$

$$\begin{aligned} \mathcal{M}_1 &= (-1, 0), \quad \mathcal{M}_2 = (1, 0), \\ \mathcal{M}_3 &= \left(0, -\sqrt{\frac{\lambda}{\mu}}\right), \quad \mathcal{M}_4 = \left(0, \sqrt{\frac{\lambda}{\mu}}\right). \end{aligned} \quad (13)$$

The orbits connecting the vacuum states can be seen on Fig. 1. Note that we can have six configurations connecting the vacua, where five are BPS states and one is non-BPS.

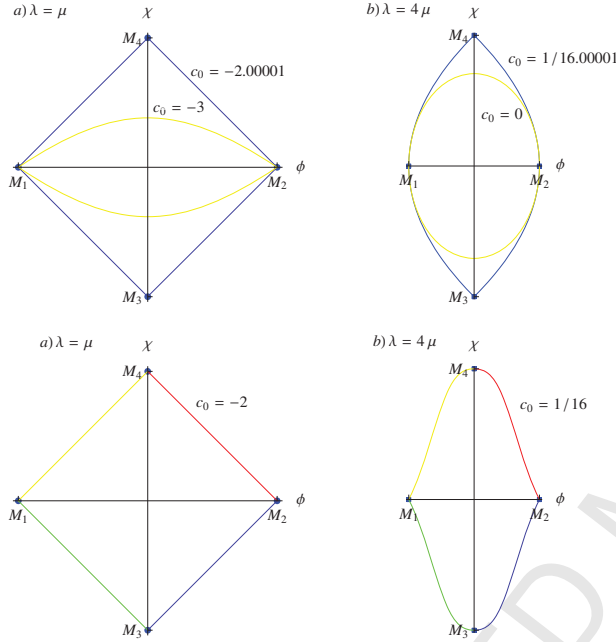


FIG. 1: Orbit for the solutions and vacuum states of the potential. The plots on the top of the Figure show the degenerate solutions and the bottom ones show the critical solutions.

Using the above results, the sectors connecting the vacua and their corresponding energies are given by

$$\begin{aligned} \mathcal{M}_1 &\rightarrow \mathcal{M}_2, \quad E_{BPS}^{(12)} = \frac{4\lambda}{3}, \\ \mathcal{M}_1 &\rightarrow \mathcal{M}_3, \quad E_{BPS}^{(13)} = \frac{2\lambda}{3}, \\ \mathcal{M}_1 &\rightarrow \mathcal{M}_4, \quad E_{BPS}^{(14)} = \frac{2\lambda}{3}, \\ \mathcal{M}_2 &\rightarrow \mathcal{M}_3, \quad E_{BPS}^{(23)} = \frac{2\lambda}{3}, \\ \mathcal{M}_2 &\rightarrow \mathcal{M}_4, \quad E_{BPS}^{(24)} = \frac{2\lambda}{3}, \\ \mathcal{M}_3 &\rightarrow \mathcal{M}_4, \quad E_{nBPS}^{(34)} = \frac{4\lambda}{3} \sqrt{\frac{\lambda}{\mu}}. \end{aligned} \quad (14)$$

Thus, we can see that four sectors have degenerate energies.

As remarked in [26], general solutions of the first-order differential equations can be found for the scalar fields, by first integrating the relation

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi} = \frac{\lambda(\phi^2 - 1) + \mu\chi^2}{2\mu\phi\chi}, \quad (15)$$

and then by rewriting one of the fields in terms of the other.

Introducing the new variable  $\rho = \phi^2 - 1$ , we can rewrite the above equation as

$$\frac{d\rho}{d\chi} - \frac{\lambda\rho}{\mu\chi} = \chi, \quad (16)$$

and the corresponding general solutions are

$$\rho(\chi) = \phi^2 - 1 = c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2, \quad (\lambda \neq 2\mu) \quad (17)$$

$$\rho(\chi) = \phi^2 - 1 = \chi^2[\ln(\chi) + c_1], \quad (\lambda = 2\mu), \quad (18)$$

where  $c_0$  and  $c_1$  are arbitrary integration constants. Substituting the above solutions in the first-order differential equation for the field  $\chi$ , we have

$$\frac{d\chi}{dr} = \pm 2\mu\chi \sqrt{1 + c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2}, \quad (\lambda \neq 2\mu) \quad (19)$$

$$\frac{d\chi}{dr} = \pm 2\mu\chi \sqrt{1 + \chi^2[\ln(\chi) + c_1]}, \quad (\lambda = 2\mu). \quad (20)$$

It has been found in Ref. [26] that in four particular cases the first equation in (19) can be solved analytically. Moreover, in order keep the solutions finite over all space,  $c_0$  cannot assume values higher than some critical ones. At the critical values, the field configuration changes drastically, as we see next.

### A. Degenerate Bloch Walls

Dutra and Hott called the first set of solutions of equation (19) degenerate Bloch walls [42] (DBW). There are two situations with exact classical solutions:

#### A1. For $c_0 < -2$ and $\lambda = \mu$

In this case we have

$$\chi_{DBW}^{(1)}(x) = \frac{2}{(\sqrt{c_0^2 - 4}) \cosh(2\mu x) - c_0}, \quad (21)$$

$$\phi_{DBW}^{(1)}(x) = \frac{(\sqrt{c_0^2 - 4}) \sinh(2\mu x)}{(\sqrt{c_0^2 - 4}) \cosh(2\mu x) - c_0}. \quad (22)$$

**A2. For  $\lambda = 4\mu$  and  $c_0 < 1/16$**

The solutions can be written as

$$\chi_{DBW}^{(2)}(x) = -\frac{2}{\sqrt{(\sqrt{1-16c_0}) \cosh(4\mu x) + 1}}, \quad (23)$$

$$\phi_{DBW}^{(2)}(x) = \frac{(\sqrt{1-16c_0}) \sinh(4\mu x)}{(\sqrt{1-16c_0}) \cosh(4\mu x) + 1}. \quad (24)$$

In Figure 2 we show some typical profiles of the DBW solutions. Note that the two-kink solution (top)  $\phi_{DBW}^{(1)}$  arises only for values of  $c_0$  close to the critical value,  $c_0^{(\text{crit})} = -2$ . For the same values of  $c_0$ , the corresponding lump-like solutions for  $\chi_{DBW}^{(1)}$  (bottom) exhibit a flat top, which disappears as we move away from  $c_0^{(\text{crit})}$ . As we will see, the related CE for these configurations carry a very distinctive signature.

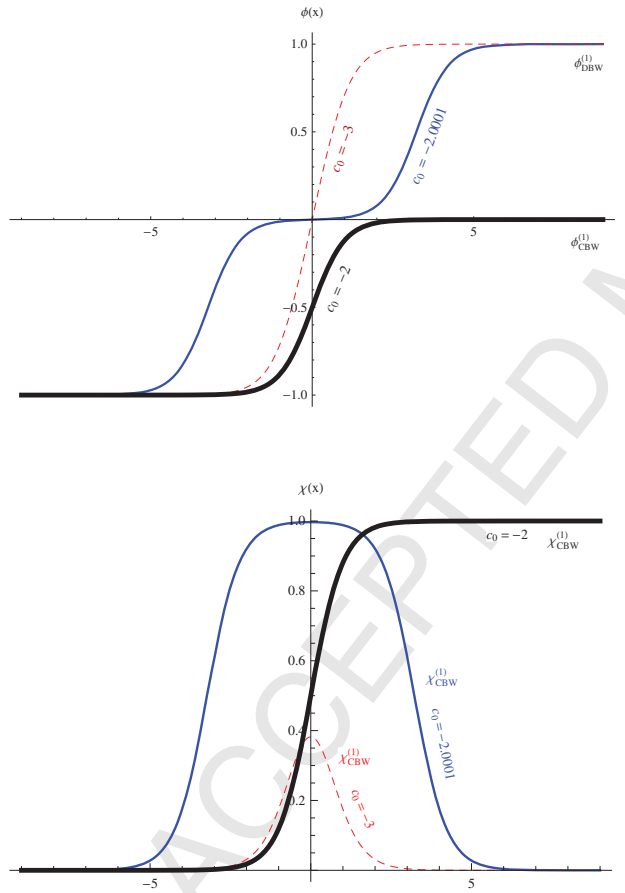


FIG. 2: Sample profiles of the DBW and CBW solutions. Thin lines (continuous and dashed) are for two DBW solutions, while the thick line is for the CBW solution.

An important feature of the DBW solutions is that their energies are degenerate with respect to  $c_0$ : for any

value of  $c_0$  the energy is given by  $E_{DBW} = 4\lambda/3$ . This means that energy alone cannot distinguish between the rich variation in the spatial profiles of the DBW solutions as  $c_0$  is varied. As we shall see, this is where the CE will play a key role.

## B. Critical Bloch Walls

An interesting class of analytical solutions, named as critical Bloch walls (CBW), was shown to exist when the constant of integration is taken to be equal to the critical value. Again, we have two cases:

**B1. For  $\lambda = \mu$  and  $c_0 = -2$**

One has the following set of solutions for the scalar fields

$$\chi_{CBW}^{(1)}(x) = \frac{1}{2} [1 \pm \tanh(\mu x)], \quad (25)$$

$$\phi_{CBW}^{(1)}(x) = -\frac{1}{2} [\tanh(\mu x) \mp 1]. \quad (26)$$

**B2. For  $\lambda = 4\mu$  and  $c_0 = 1/16$**

Now, the solutions for the fields are given by

$$\chi_{CBW}^{(2)}(x) = \sqrt{2} \frac{\cosh(\mu x) \pm \sinh(\mu x)}{\sqrt{\cosh(2\mu x)}}, \quad (27)$$

$$\phi_{CBW}^{(2)}(x) = \frac{1}{2} [\pm 1 - \tanh(2\mu x)]. \quad (28)$$

In Fig. 2 we also show the case CBW for  $c_0 = -2$ . Now, the energy is  $E_{CBW} = 2\lambda/3$ , consistent with the energy for the DBW case since one can use two CBW configurations in order to connect the vacua connected by the DBW:  $E_{DBW} = 4\lambda/3 = 2 E_{CBW}$ , as can be seen from Fig. 1.

## 3. CONFIGURATIONAL ENTROPY FOR TWO INTERACTING SCALAR FIELDS

Recently, GS showed that a configurational entropy measure in functional space can be used to discriminate between same-energy spatially-localized solutions [27]. The configurational entropy (CE) is defined as

$$S_c[f] = - \int d^d \mathbf{k} \tilde{f}(\mathbf{k}) \ln[\tilde{f}(\mathbf{k})], \quad (29)$$

where  $d$  is the number of spatial dimensions, and  $\tilde{f}(\mathbf{k}) = f(\mathbf{k})/f_{\text{max}}(\mathbf{k})$ . The function  $f(\mathbf{k})$  is defined as the modal

fraction

$$f(\mathbf{k}) = \frac{|F(\mathbf{k})|^2}{\int d^d \mathbf{k} |F(\mathbf{k})|^2}. \quad (30)$$

$f_{\max}(\mathbf{k})$  is the maximal modal fraction, that is, the mode giving the highest contribution. This normalization guarantees that  $S_c[f]$  is positive-definite. The function  $F(\mathbf{k})$  represents the Fourier transform of the energy density of the configuration. In order to compute the CE the energy density must be square-integrable even if the fields are not.

Here, we will extend the procedure presented in [27] to models with two coupled fields. For static configurations of two interacting real scalar fields the energy density is written as

$$\rho(x) = \frac{1}{2} \left[ (\partial_x \phi)^2 + (\partial_x \chi)^2 + V(\phi, \chi) \right]. \quad (31)$$

Following the approach presented in [27], we use the energy density of the DBW and CBW configurations to compute their related CE. The Fourier transform is given by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ikx} \rho(x), \quad (32)$$

Plancherel's theorem states that

$$\int_{-\infty}^{\infty} dx |\rho(x)|^2 = \int_{-\infty}^{\infty} dk |F(k)|^2. \quad (33)$$

Again, we stress that the spatially-localized energy densities must be square-integrable bounded functions  $\rho(x) \in L^2(\mathbf{R})$ .

#### 4. ENTROPY FOR DBW AND CBW

We now use the approach presented in the previous section to obtain the configurational entropy of the DBW and the CBW configurations. Let us begin with the DBW case, which has two sets of exact solutions, given in section II.A. Starting with case A1,  $\lambda = \mu$  and  $c_0 < -2$ , from the scalar fields given in equations (21) and (22), we obtain the corresponding energy density as

$$\begin{aligned} \rho_{DBW}^{(1)}(x) &= \frac{6\mu^2}{[a_1 + \cosh(2\mu x)]^4} \\ &\quad - \frac{8\mu^2 c_0 \cosh(2\mu x)}{\alpha [a_1 + \cosh(2\mu x)]^4} + \frac{2\mu^2 \cosh(4\mu x)}{[a_1 + \cosh(2\mu x)]^4}, \end{aligned} \quad (34)$$

where  $\alpha = \alpha(c_0) \equiv c_0/\sqrt{c_0^2 - 4}$  and  $a_1 \equiv -c_0/\alpha$ .

On the other hand, for case A2, where  $\lambda = 4\mu$  and  $c_0 < 1/16$ , the corresponding energy density is

$$\begin{aligned} \rho_{DBW}^{(2)}(x) &= -\frac{16\mu^2 [\beta^2 + \cosh(2\mu x)]}{\beta^2 [a_2 + \cosh(4\mu x)]^4} \\ &\quad - \frac{4\mu^2 [7 \cosh(4\mu x) + \cosh(12\mu x)]}{\beta [a_2 + \cosh(4\mu x)]^4}, \end{aligned} \quad (35)$$

with  $\beta = \beta(c_0) \equiv \sqrt{1 - 16c_0}$  and  $a_2 \equiv 1/\beta$ .

At this point, we note again that different solutions to the DBW equations have energies that are degenerate with respect to  $c_0$ . In [27], Gleiser and Stamatopoulos studied a case where different trial functions used to approximate the actual solution were energy-degenerate. They showed that the CE could be used to select which of the trial functions was a better fit to the exact solution: that which had minimal CE. Here, we have a novel situation where the actual analytical solutions to the equations of motion have an infinite degeneracy with respect to a single parameter ( $c_0$ ). We will follow the approach in GS and examine whether the CE can be used to discriminate between solutions which are energy-degenerate. In this way, we are proposing that the configurational entropy is an excellent tool to resolve ambiguous situations that may emerge from Hamilton's variational principle.

We thus proceed to compute the Fourier transform of the energy density (34) and (35), which gives the modal fraction (30). Using (32), we have

$$F^{(1)}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ikx} \rho_{DBW}^{(1)}(x), \quad (36)$$

$$F^{(2)}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ikx} \rho_{DBW}^{(2)}(x). \quad (37)$$

In order to obtain an analytical expression for the above Fourier transforms, it is useful to introduce the generalized integral

$$I^{(n)}(a_n, \gamma, \delta, k) = \int_{-\infty}^{\infty} dx \frac{e^{ikx} \cosh(\gamma x)}{[a_n + \cosh(\delta x)]^4}. \quad (38)$$

After lengthy but straightforward calculations, one finds

$$I^{(n)}(a_n, \gamma, \delta, k) = \frac{8}{\delta} \sum_{j=1}^2 G_j(a_n, \gamma, \delta, k), \quad (39)$$

where

$$\begin{aligned} G_j(a_n, \gamma, \delta, k) &= \frac{1}{\Omega_j + 4} F_1[A_j; B_j, B'_j; C_j; X_n, Y_n] \\ &\quad - \frac{1}{\Omega_j - 4} F_1[\bar{A}_j; \bar{B}_j, \bar{B}'_j; \bar{C}_j; X_n, Y_n], \end{aligned} \quad (40)$$

and the functions  $F_1[A_j; B_j, B'_j; C_j; X_n, Y_n]$  and  $F_1[\bar{A}_j; \bar{B}_j, \bar{B}'_j; \bar{C}_j; X_n, Y_n]$  are the so-called Appell



hypergeometric functions of two variables with

$$\Omega_j = i\omega + (-1)^{j+1}\Omega, \quad \omega = k/\delta, \quad \Omega = \gamma/\delta,$$

$$A_j = \Omega_j + 4, \quad B_j = B'_j = 4, \quad C_j = \Omega_j + 5,$$

$$\bar{A}_j = -\Omega_j + 4, \quad \bar{B}_j = \bar{B}'_j = 4, \quad \bar{C}_j = -\Omega_j + 5,$$

$$X_n = -1/\left[a_n - \sqrt{a_n^2 - 1}\right],$$

$$Y_n = -1/\left[a_n + \sqrt{a_n^2 - 1}\right].$$

We can now write the Fourier transforms of (36) and (37) in the following compact forms

$$F^{(1)}(k) = \frac{2\mu^2}{\sqrt{2\pi}} \left[ 3I^{(1)}(a_1, 0, 2\mu, k) - \frac{4c_0}{\alpha} I^{(1)}(a_1, 2\mu, 2\mu, k) + I^{(1)}(a_1, 4\mu, 2\mu, k) \right], \quad (41)$$

$$F^{(2)}(k) = -\frac{4\mu^2}{\sqrt{2\pi}} \left[ I^{(2)}(a_2, 0, 4\mu, k) + \frac{4}{\beta^2} I^{(2)}(a_2, 2\mu, 4\mu, k) + \frac{7}{\beta} I^{(2)}(a_2, 4\mu, 4\mu, k) + \frac{1}{\beta} I^{(2)}(a_2, 12\mu, 4\mu, k) \right]. \quad (42)$$

In order to obtain the modal fraction (30) it is necessary to evaluate (33), clearly a daunting task. To proceed analytically, we evaluate the integrals for  $F^{(1)}(k)$  and  $F^{(2)}(k)$  numerically, and fit them as functions of the single parameter  $c_0$  as

$$\int_{-\infty}^{\infty} dk \left| F^{(1)}(k) \right|^2 \simeq g_1 - g_2 e^{g_3 c_0}, \quad (43)$$

$$\int_{-\infty}^{\infty} dk \left| F^{(2)}(k) \right|^2 \simeq h_1 - h_2 e^{h_3 c_0}, \quad (44)$$

where  $g_1 = 0.8481$ ,  $g_2 = 3.8834$ ,  $g_3 = 1.1332$ ,  $h_1 = 41.0711$ ,  $h_2 = 23.2854$  and  $h_3 = 1.1699$ .

The modal fractions can be approximated by

$$f^{(1)}(k) \simeq \frac{|F^{(1)}(k)|^2}{g_1 - g_2 e^{g_3 c_0}}, \quad f^{(2)}(k) \simeq \frac{|F^{(2)}(k)|^2}{h_1 - h_2 e^{h_3 c_0}}. \quad (45)$$

These modal fractions are plotted in Fig. 3. As can be seen, they are localized and exhibit a maximum at  $k = 0$ . These expressions are to be used into equation (29) in order to obtain the CE for each of the two DBW cases:

$$S_c^{(1)} \simeq - \int dk \tilde{f}^{(1)}(k) \ln[\tilde{f}^{(1)}(k)], \quad (46)$$

$$S_c^{(2)} \simeq - \int dk \tilde{f}^{(2)}(k) \ln[\tilde{f}^{(2)}(k)]. \quad (47)$$

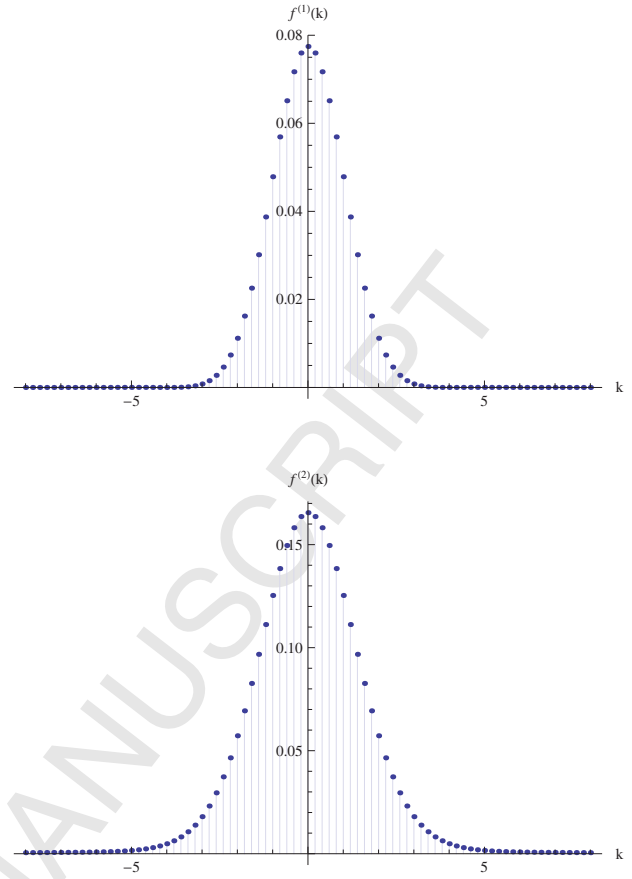


FIG. 3: Modal fractions with  $\mu = 1$ . Note that the maximum is at  $k = 0$ .

To compute the configurational entropy, we must integrate equations (47) numerically. The results are shown in Fig. 4, where the CE is plotted as a function of the parameter  $c_0$ . It is quite remarkable that the CE shows such rich structure for varying  $c_0$  while the energies for all these configurations are simply degenerate. There is a sharp minimum at the value  $c_0^{(\min)} \simeq -2.005$ , the region of parameter space where the double-kink solution is most prominent, within our numerical accuracy. This can be seen by plotting the three inflection points of the solution for  $\phi_{\text{DBW}}^{(1)}$ , and showing how they progressively merge into a single inflection point—a single kink—as  $c_0$  is decreased. Below  $c_0^{(\text{trans})}$ , an inflection point for CE, the field configurations undergo a quick transition, where the two-kink solution in the field  $\phi_{\text{DBW}}^{(1)}$  rapidly converges into a single kink at  $c_0 \lesssim -2.30$ , while the lump-like solutions for  $\chi_{\text{DBW}}^{(1)}$ , which have a flat-top profile for  $c_0 > c_0^{(\text{trans})}$ , become more Gaussian-like.

For completeness, we examined how the results vary with respect to the two-field coupling constant  $\lambda$ , taken to be  $\lambda = 1$  in Fig. 4. The qualitative features remain the same, at least for these values of  $\lambda$ . The results are shown in Table I:



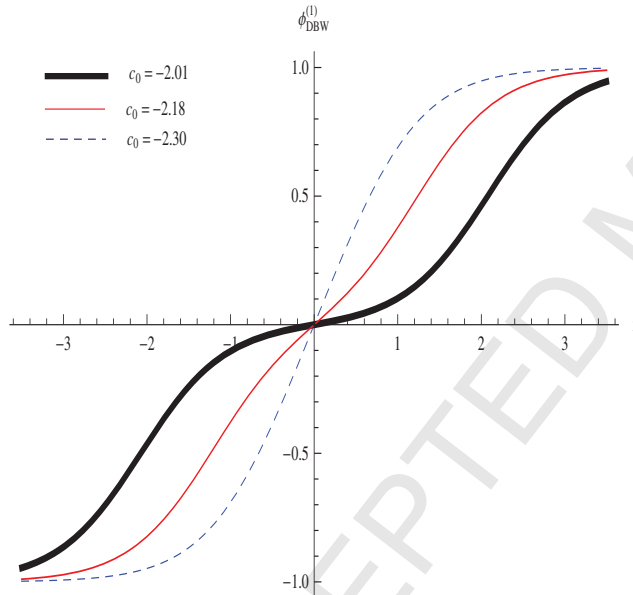
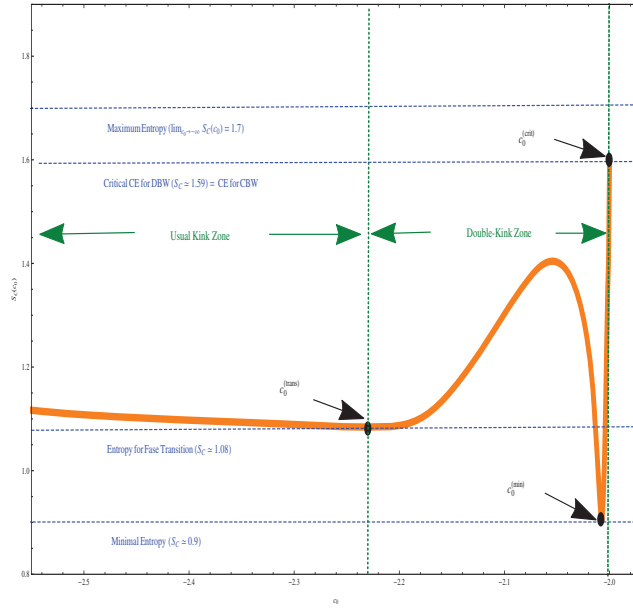


FIG. 4: The configurational entropy (top) for DBW solutions as a function of the parameter  $c_0$  and a few of its corresponding  $\phi_{\text{DBW}}^{(1)}$ -field configurations (bottom). The curves correspond to choosing  $\lambda = 1$ .

## 5. CONCLUSIONS.

We investigated the properties of kink-like solutions in a class of interacting two-field models in two space-time dimensions. The models we considered are especially interesting because we can use the BPS approach to find exact analytical solutions. In particular, a class

$\lambda$	$c_0^{(\text{crit})}$	$S_c^{(\text{crit})}$	$c_0^{(\text{min})}$	$S_c^{(\text{min})}$	$c_0^{(\text{trans})}$	$S_c^{(\text{trans})}$	$\text{Lim}_{c_0 \rightarrow \infty} S_c$
$\lambda = 1$	-2.00000	1.59	-2.0075	0.9049	-2.2303	1.0838	1.7
$\lambda = 1.4$	-2.63384	1.45	-2.6361	1.0949	-2.9735	1.2465	2.2
$\lambda = 2.2$	4.20083	2.02	4.2006	1.2857	4.0248	1.3606	2.3

TABLE I: Configurational entropy and values of corresponding  $c_0$  for various choices of the two-field coupling  $\lambda$  for Degenerate Bloch Walls.

of these solutions known as Degenerate Bloch Walls is degenerate in energy, even if their spatial profiles are widely different. Using the configurational entropy measure of Gleiser and Stamatopoulos, we were able to extract information about the different solutions which is clearly related to their spatial profiles. We thus propose that this information-entropic measure is an essential tool in the study of complex spatially-localized configurations, providing valuable information beyond simple energetics.

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